M12/5/MATHL/HP2/ENG/TZ1/XX/M

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# MARKSCHEME

## May 2012

## MATHEMATICS

## Higher Level

## Paper 2

15 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**; working must be seen.

***(M)*** Marks awarded for **Method**; may be implied by **correct** subsequent working.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***(A)*** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

***R*** Marks awarded for clear **Reasoning**.

***N*** Marks awarded for **correct** answers if **no** working shown.

***AG*** Answer given in the question and so no marks are awarded.

### Using the markscheme

#### 1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2012**". It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

**3 *N* marks**

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

**4 Implied marks**

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

**5 Follow through marks**

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

**6 Mis-read**

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

**7 Discretionary marks *(d)***

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

**8 Alternative methods**

*Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.*

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

**9 Alternative forms**

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  **Example**: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

**10 Accuracy of Answers**

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

**11 Crossed out work**

*If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.*

## 12 Calculators

*A GDC is required for paper 2, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.*

**Calculator notation**

The Mathematics HL guide says:

*Students must always use correct mathematical notation, not calculator notation.*

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

## 13 More than one solution

*Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.*

## SECTION A

**1.** $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + k$ ***M1A1***

For use of discriminant $b^2 - 4ac = 0$ or completing the square $3(x-2)^2 + k - 12$ (***M1)***

$144 - 12k = 0$ ***(A1)***

**Note:** Accept trial and error, sketches of parabolas with vertex (2,0) or use of second derivative.

$k = 12$ ***A1***

***[5 marks]***

**2.** $k\int_1^2 2^{\frac{1}{x}}\,\mathrm{d}x = 1 \Rightarrow k = \frac{1}{\int_1^2 2^{\frac{1}{x}}\,\mathrm{d}x} \quad (= 0.61556\ldots)$ ***(M1)(A1)***

$\mathrm{E}(X) = k\int_1^2 x2^{\frac{1}{x}}\,\mathrm{d}x = 2.39....k$ or 1.47 ***M1A1***

**Note:** Condone missing d$x$ in any part of the question.

***[4 marks]***

**3.** (a) $\binom{10}{6} = 210$ ***(M1)A1***

***[2 marks]***

(b) $2 \times \binom{8}{5} = 112$ ***(M1)A1A1***

**Note:** Accept 210 – 28 – 70 = 112

***[3 marks]***

(c) $\frac{112}{210} \left(= \frac{8}{15} = 0.533\right)$ ***(M1)A1***

***[2 marks]***

***Total [7 marks]***

**4.** point on line is $x = \frac{-1-5\lambda}{5}$, $y = \frac{9+5\lambda}{5}$, $z = \lambda$ or similar ***M1A1***

**Note:** Accept use of point on the line or elimination of one of the variables using the equations of the planes

$\frac{-1-5\lambda}{5} - \frac{9+5\lambda}{5} + 2\lambda = k$ ***M1A1***

**Note:** Award ***M1A1*** if coordinates of point and equation of a plane is used to obtain linear equation in *k* or equations of the line are used in combination with equation obtained by elimination to get linear equation in *k*.

$k = -2$ ***A1***

***[5 marks]***

**5.** (a) 50 ***A1***

***[1 mark]***

(b) Lower quartile is 4 so at least 26 obtained a 4 ***R1***
Lower bound is 26 ***A1***

Minimum is 2 but the rest could be 4 ***R1***

So upper bound is 49 ***A1***

**Note:** Do not allow follow through for ***A*** marks.

**Note:** If answers are incorrect award ***R0A0***; if argument is correct but no clear lower/upper bound is stated award ***R1A0;*** award ***R0A1*** for correct answer without explanation or incorrect explanation.

***[4 marks]***

***Total [5 marks]***

**6.** $h(x) = f(x-3) - 2 = \ln(x-3) - 2$ ***(M1)(A1)***
$g(x) = -h(x) = 2 - \ln(x-3)$ ***M1***

**Note:** Award **M1** only if it is clear the effect of the reflection in the *x*-axis:
the expression is correct ***OR***
there is a change of signs of the previous expression ***OR***
there's a graph or an explanation making it explicit

$= \ln e^2 - \ln(x-3)$ ***M1***

$= \ln\left(\frac{e^2}{x-3}\right)$ ***A1***

***[5 marks]***

**7.**

$X \sim \text{Po}(m)$

$\text{P}(X = 2) = \text{P}(X < 2)$ ***(M1)***

$\frac{1}{2}m^2 \text{e}^{-m} = \text{e}^{-m}(1+m)$ ***(A1)(A1)***

$m = 2.73 \quad \left(1+\sqrt{3}\right)$ ***A1***

in four hours the expected value is 10.9 $\left(4+4\sqrt{3}\right)$ ***A1***

**Note:** Value of *m* does not need to be rounded.

***[5 marks]***

**8.** $x = r - \frac{r}{h}y$ or $x = \frac{r}{h}(h-y)$ (or equivalent) ***(A1)***

$\int \pi x^2 \, \text{d}y$

$= \pi \int_0^h \left(r - \frac{r}{h}y\right)^2 \text{d}y$ ***M1A1***

**Note:** Award ***M1*** for $\int x^2 \, \text{d}y$ and ***A1*** for correct expression.

Accept $\pi \int_0^h \left(\frac{r}{h}y - r\right)^2 \text{d}y$ and $\pi \int_0^h \left(\pm\left(r - \frac{r}{h}x\right)\right)^2 \text{d}x$

$= \pi \int_0^h \left(r^2 - \frac{2r^2}{h}y + \frac{r^2}{h^2}y^2\right) \text{d}y$ ***A1***

**Note:** Accept substitution method and apply markscheme to corresponding steps.

$= \pi \left[r^2 y - \frac{r^2 y^2}{h} + \frac{r^2 y^3}{3h^2}\right]_0^h$ ***M1A1***

**Note:** Award ***M1*** for attempted integration of any quadratic trinomial.

$= \pi \left(r^2 h - r^2 h + \frac{1}{3}r^2 h\right)$ ***M1A1***

**Note:** Award ***M1*** for attempted substitution of limits in a trinomial.

$= \frac{1}{3}\pi r^2 h$ ***A1***

**Note:** Throughout the question do not penalize missing d*x*/d*y* as long as the integrations are done with respect to correct variable

***[9 marks]***

**9.** $\left(x-\frac{2}{x}\right)^4 = x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ ***(M1)(A1)***

$\left(x^2+\frac{2}{x}\right)^3 = x^6 + 6x^3 + 12 + \frac{8}{x^3}$ ***(M1)(A1)***

| **Note:** | Accept unsimplified or uncalculated coefficients in the constant term |
|---|---|

$= 24 \times 12$ ***(M1)(A1)***

$= 288$ ***A1***

***[7 marks]***

**10.** attempt to find intersections ***M1***

intersections are $\left(\frac{10}{m+2}, \frac{10m}{m+2}\right)$ and $\left(\frac{10m}{2m-1}, -\frac{10}{2m-1}\right)$ ***A1A1***

area of triangle $= \frac{1}{2} \times \frac{\sqrt{100+100m^2}}{(m+2)} \times \frac{\sqrt{100+100m^2}}{(2m-1)}$ ***M1A1A1***

$= \frac{50(1+m^2)}{(m+2)(2m-1)}$

minimum when $m = 3$ ***(M1)A1***

***[8 marks]***

## SECTION B

**11.** (a) $(3.79, -5)$ ***A1***

***[1 mark]***

(b) $p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$ ***A1A1***

***[2 marks]***

(c) $f'(x) = 3\cos x - 4\sin x$ ***(M1)(A1)***

$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43\ldots$ ***(A1)***

$(y = -4)$ ***A1***

Coordinates are $(4.43, -4)$

***[4 marks]***

(d) $m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}$ ***(M1)***

gradient at P is $-4$ so gradient of normal at P is $\frac{1}{4}$ ***(A1)***

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$ ***(A1)***

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570\ldots)$ $\left(\text{or } y = 0.25x + 2.60...\right)$ ***(M1)***

equation of normal at Q is $y - 3 = -\frac{1}{4}\left(x - 5.999\ldots\right)$ $\left(\text{or } y = -0.25x + \underbrace{4.499...}\right)$ ***(M1)***

**Note:** Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where $(a, b)$ are coordinates of P and Q (or in $y = mx + c$ with $c$ determined using coordinates of Pand Q.

intersect at $(3.79, 3.55)$ ***A1A1***

**Note:** Award ***N2*** for 3.79 without other working.

***[7 marks]***

***Total [14 marks]***

**12.** (a) (i) $X \sim \text{Po}(11)$ ***(M1)***

$\text{P}(X \leq 11) = 0.579$ ***(M1)A1***

(ii) $\text{P}(X > 8 \mid X < 12) =$ ***(M1)***

$= \dfrac{\text{P}(8 < X < 12)}{\text{P}(X < 12)} \quad \left( \text{or } \dfrac{\text{P}(X \leq 11) - \text{P}(X \leq 8)}{\text{P}(X \leq 11)} \text{ or } \dfrac{0.3472...}{0.5792...} \right)$ ***A1***

$= 0.600$ ***A1*** ***N2***

***[6 marks]***

(b) (i) $Y \sim \text{Po}(m)$

$\text{P}(Y > 3) = 0.24$ ***(M1)***

$\text{P}(Y \leq 3) = 0.76$ ***(M1)***

$\text{e}^{-m}\left(1 + m + \dfrac{1}{2}m^2 + \dfrac{1}{6}m^3\right) = 0.76$ ***(A1)***

**Note:** At most two of the above lines can be implied.

Attempt to solve equation with GDC ***(M1)***

$m = 2.49$ ***A1***

(ii) $A \sim \text{Po}(4.98)$

$\text{P}(A > 5) = 1 - \text{P}(A \leq 5) = 0.380\ldots$ ***M1A1***

$W \sim \text{B}(4, 0.380\ldots)$ ***(M1)***

$\text{P}(W \geq 2) = 1 - \text{P}(W \leq 1) = 0.490$ ***M1A1***

***[10 marks]***

(c) $\text{P}(A < 25) = 0.8$, $\text{P}(A < 18) = 0.4$

$\dfrac{25 - \mu}{\sigma} = 0.8416\ldots$ ***(M1)(A1)***

$\dfrac{18 - \mu}{\sigma} = -0.2533\ldots$ (or $-0.2534$ from tables) ***(M1)(A1)***

solving these equations ***(M1)***

$\mu = 19.6$ ***A1***

**Note:** Accept just 19.6, 19 or 20; award A0 to any other final answer.

***[6 marks]***

***Total [22 marks]***

**13.** (a) $\vec{AB} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \Rightarrow AB = \sqrt{72}$ ***A1***

$\vec{AC} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \Rightarrow AC = \sqrt{72}$ ***A1***

so they are the same ***AG***

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 36 = \left(\sqrt{72}\right)\left(\sqrt{72}\right)\cos\theta$ ***(M1)***

$\cos\theta = \dfrac{36}{\left(\sqrt{72}\right)\left(\sqrt{72}\right)} = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$ ***A1AG***

**Note:** Award ***M1A1*** if candidates find BC and claim that triangle ABC is equilateral.

***[4 marks]***

(b) **METHOD 1**

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 6 & -6 \\ -6 & 0 & -6 \end{vmatrix} = -36\boldsymbol{i} + 36\boldsymbol{j} + 36\boldsymbol{k}$ ***(M1)A1***

equation of plane is $x - y - z = k$ ***(M1)***

goes through A, B or C $\Rightarrow x - y - z = 2$ ***A1***

***[4 marks]***

**METHOD 2**

$x + by + cz = d$ (or similar) ***M1***

$5 - 2b + 5c = d$

$5 + 4b - c = d$ ***A1***

$-1 - 2b - c = d$

solving simultaneously ***M1***

$b = -1,\ c = -1,\ d = 2$

so $x - y - z = 2$ ***A1***

***[4 marks]***

(c) (i) midpoint is $(5, 1, 2)$, so equation of $\Pi_1$ is $y - z = -1$ ***A1A1***

(ii) midpoint is $(2, -2, 2)$, so equation of $\Pi_2$ is $x + z = 4$ ***A1A1***

**Note:** In each part, award ***A1*** for midpoint and ***A1*** for the equation of the plane.

***[4 marks]***

*continued ...*

*Question 12 continued ...*

(d) **EITHER**

solving the two equations above ***M1***

$L: \boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ***A1***

**OR**

L has the direction of the vector product of the normal vectors to the planes $\Pi_1$ and $\Pi_2$ ***(M1)***

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$

(or its opposite) ***A1***

**THEN**

direction is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ as required ***R1***

***[3 marks]***

(e) D is of the form $(4-\lambda, -1+\lambda, \lambda)$ ***M1***

$(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = 72$ ***M1***

$3\lambda^2 - 6\lambda - 45 = 0$

$\lambda = 5$ or $\lambda = -3$ ***A1***

D(–1, 4, 5) ***AG***

| | |
|---|---|
| ***Note:*** | Award ***M0M0A0*** if candidates just show that D(–1, 4, 5) satisfies AB=AD; |
| | Award ***M1M1A0*** if candidates also show that D is of the form $(4-\lambda, -1+\lambda, \lambda)$ |

***[3 marks]***

*continued ...*

*Question 12 continued ...*

(f) **EITHER**

G is of the form $(4-\lambda, -1+\lambda, \lambda)$ and $DG = AG$, BG or CG ***M1***

*e.g.* $(1+\lambda)^2 + (-1-\lambda)^2 + (5-\lambda)^2 = (5-\lambda)^2 + (5-\lambda)^2 + (5-\lambda)^2$ ***M1***

$(1+\lambda)^2 = (5-\lambda)^2$

$\lambda = 2$ ***A1***

$G(2, 1, 2)$ ***AG***

**OR**

G is the centre of mass (barycentre) of the regular tetrahedron ABCD ***(M1)***

$$G\left(\frac{5+5+(-1)+(-1)}{4}, \frac{-2+4+(-2)+4}{4}, \frac{5+(-1)+(-1)+5}{4}\right)$$ ***M1A1***

**THEN**

**Note:** the following part is independent of previous work and candidates may use ***AG*** to answer it (here it is possible to award ***M0M0A0A1M1A1***)

$$\vec{GD} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} \text{ and } \vec{GA} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$ ***A1***

$$\cos\theta = \frac{-9}{\left(3\sqrt{3}\right)\left(3\sqrt{3}\right)} = -\frac{1}{3} \Rightarrow \theta = 109^\circ \text{ (or 1.91 radians)}$$ ***M1A1***

***[6 marks]***

***Total [24 marks]***